JOURNAL OF ALGEBRA 17, 273–298 (1971)

Rings of Quotients and Morita Contexts*

S. A. AMITSUR

University of California, Los Angeles and
The Hebrew University, Jerusalem

Communicated by I. N. Herstein

Received February 24, 1970

The Morita context which has been introduced in [9] was used since to prove Wedderburn theorem on the structure of simple rings [1] and in a disguised form it has been the basic tool in the study of primitive rings with a minimal left ideal (e.g. [6, p. 75]). Other applications, though not stated in an explicit form, can be found in various places. In the present paper we use the Morita context, to obtain various results: Goldie's theorem [2, 3] on the ring of quotients of semi-prime rings, following some ideas of Procesi (e.g. [5, p. 66]) and as a speciliazation one obtains Wedderburn's structure theorems of semi-simple artinian rings. The same methods are very useful in studying the ring of endomorphisms of modules V and in particular we obtain Zelmanowitz result [14] on the structure of Hom(V, V), if V is torsionless and finitely generated, and R is prime or semi-prime. Also, obtained are the facts that ring of endomorphisms of primitive rings are primitive [4, 10] and information on the radicals of ring of endomorphisms.

I. Preliminary Remarks

A Morita context (m. c.) will be a set $M = (R, V, W, S)$ and two maps $(\,,\,)$, $[\,,\,]$, where R and S are (associative) rings; $V = {}_RV_S$ is an $R - S$ bimodule — a left R-module and a right S-module, and $W = {}_SW_R$ is an $S - R$ bimodule. The map $(\,,\,) : V \otimes_S W \to R$ is an $R - R$ bilinear map and $[\,,\,] : W \otimes_R V \to S$ is $S - S$ bilinear, furthermore these two maps satisfy the associativity conditions:

$$1_V \otimes [\ ,\ ] = (\ ,\ ) \otimes 1_V : V \underset{S}{\otimes} W \underset{R}{\otimes} V \to V;$$

$$[\ ,\ ] \otimes 1_W = 1_W \otimes (\ ,\ ) : W \underset{R}{\otimes} V \underset{S}{\otimes} W \to W.$$

* The Preparation of this paper was sponsored in part by NSF Grant #GP-9661.

For further reference we list the requirements imposed on the rings, modules and the bilinear maps ( ), [ ] so to form a Morita context: For every $r \in R$, $s \in S$, $v, v_i \in V$ and $w, w_i \in W$:

$$r(vs) = (rv)s; \qquad s(wr) = (sw)r; \tag{1}$$

$$(v_1 + v_2, w) = (v_1, w) + (v_2, w); \qquad (v, w_1 + w_2) = (v, w_1) + (v, w_2); \tag{2}$$

$$r(v, w) = (rv, w); \qquad (v, w)r = (v, wr); \tag{3}$$

$$(vs, w) = (v, sw); \tag{4}$$

$$[w_1 + w_2, v] = [w_1, v] + [w_2, v]; \qquad [w, v_1 + v_2] = [w, v_1] + [w, v_2]; \tag{5}$$

$$s[w, v] = [sw, v]; \qquad [w, v]s = [w, vs]; \tag{6}$$

$$[wr, v] = [w, rv]; \tag{7}$$

and finally the associativity conditions

$$v_1[w, v] = (v_1, w)v; \qquad [w, v]\, w_1 = w(v, w_1). \tag{8}$$

Note that by the linearity and associativity conditions we also have for every linearly closed subsets $V_1, V_2 \subseteq V$, $W_1, W_2 \subseteq W$:

$$V_1[W_1, V_2] = (V_1, W_1)\, V_2; \qquad [W_1, V_1]\, W_2 = W_1(V_1, W_2), \tag{8'}$$

where (V', W') denotes the set of all finite sums $\sum (v_i', w_i')$ where v_i' ranges over all V' a linearly closed subset of V and w_i' ranges over all $w' \in W'$ a linearly closed subset of W. Furthermore, if W' is an R-submodule of W then (V', W') is a right ideal in R, and if V' is an R-submodule then (V', W') is a left ideal. Similar results hold for $[W', V']$.

Morita contexts are in abundance in associative rings, and probably the most general one is the following:

Let $V = {}_RV$ be a left R-module, and let $*V = \mathrm{Hom}_R(V, R)$ and $\mathscr{E} = \mathrm{Hom}_R(V, V)$. In the following as well as in the rest of the paper we shall write maps and endomorphisms on the opposite side of the operators with which they commute; thus, for left module ${}_RV$, the endomorphism of $\mathrm{Hom}_R(V, X)$ will be written on the right of the elements on which they act; and for right modules V_S the maps will be written on the left. But when both sides are involved or the maps are ring homomorphisms they will be written on the top.

In the example we are dealing $M = (R, V, *V, \mathscr{E})$ (e.g. [13]) the given maps: $(,) : V \otimes *V \to R$ is given by $(v, \varphi) = (v)\varphi$ the evaluation map, i.e. (v, φ) is the value of φ at v; and $[,] : *V \otimes V \to \mathscr{E}$ is given by $(*)[\varphi, v] = (*, \varphi)v$.

Another important example is a generalization of an example used by Kaplansky for obtaining the structure theory of primitive rings with a minimal left ideal, and later (e.g. [6, p. 75]) by Bass to obtain the Wedderburn theorem for simple Artinian rings [1] and a similar one will be used here to obtain Goldie's theorem as well as their results. The example is the following: Let ${}_RL$, I_R be a left and right ideals in a ring R, and S be any subring $IL \subseteq S \subseteq L \cap I$; then we have a m. c. $M = (R, L, I, S)$ where the maps $(\, , \,)$, $[\, , \,]$ are the ordinary multiplication in R, and the associativities requirements for this m.c. is the associative law for multiplication in R. The following special case $L = Re$, $I = eR$, $S = eRe$ where e is a primitive idempotent was the one used by Kaplansky and Bass.

One can define a map $M \to M_0$ between two m.c. $M = (R, V, W, S)$ and $M_0 = (R_0, V_0, W_0, S_0)$ to be a set of four homomorphisms $\tau = (\tau_R, \tau_V, \tau_W, \tau_S)$, where $\tau_R : R \to R_0$; $\tau_S : S \to S_0$ are ring homomorphisms $\tau_V : V \to V_0$, $\tau_W : W \to W_0$ are module homomorphisms commuting with τ_R, τ_S, and finally: $(v, w)^\tau = (v^\tau, w^\tau)$ and $[w, v]^\tau = [w^\tau, v^\tau]$ (where we used τ without indices to simplify notations).

We shall need the following known notions:

(1) Let ${}_RU$ be a R-module, then $d({}_RU)$-the *dimension* of U as a left R-module is the supremum of the number of summands whose direct sum $U_1 \oplus \cdots \oplus U_t$ is a submodule of U.

(2) In particular ${}_RU$ will be said to be *uniform* if $d({}_RU) = 1$, i.e. for every U_1, U_2 non zero submodule of U, $U_1 \cap U_2 \neq 0$.

(3) A (left-) *Öre domain* is a ring R without zero divisors and such that $d({}_RR) = 1$, which is equivalent to saying that for every non zero $a, b \in R$ there exist x, y such that $xa = yb \neq 0$. Recall, that any left öre domain R has a left ring of quotient $\bar{R} = \{a^{-1}b \mid a \neq 0, b \in R\}$ and every left R-module ${}_RV$ is uniquely embedded in a left $\bar{R}$ module $\bar{V} = \{a^{-1}v \mid a \neq a \in R, v \in V\}$ which is canonically isomorphic with $\bar{R} \otimes_R V$.

(4) A submodule ${}_RU \subseteq {}_RV$ is *large* (or *essential*) in V if $U \cap N \neq 0$ for every non zero submodule N in V. The lattice of all large left ideals in R play an important role in the structure of general ring of quotients; and one defines for every R-module V, the *(ℓ-) singular* submodule $Z({}_RV) = \{v \mid (0 : v)$ is a large left ideal in $R\}$ where $(0 : v) = \{r \mid r \in R, rv = 0\}$. $Z({}_RR)$ is a two sided ideal in R known as the *left singular* ideal of R.

Note that $v \in Z({}_RV)$ iff for every ${}_RL \neq 0$ there exist $xv = 0$, $0 \neq x \in L$.

(5) A module ${}_RV$ is *torsionless* if $\forall v \neq 0$ in V there exists $\varphi \in \mathrm{Hom}_R(V, R)$ such that $(v, \varphi) \neq 0$. A submodule $W_0 \subseteq \mathrm{Hom}_R(V, R) = {}^*V$ will be called a *total submodule* of ${}^*V$, if $\forall v \neq 0$ in V there exists a $\varphi \in W_0$ such that $(v, \varphi) \neq 0$. Thus V is torsionless if ${}^*V$ is a total submodule of itself, and in fact this condition is necessary and sufficient.

II. Rings with Uniform Modules V and Total Submodules

Our first result is elementary:

Theorem 1. *A ring R and a module ${}_RV$ are part of a m. c. $M = (R, V, W, S)$ such that the map $(\, , \,)$ satisfies: $(v, W) = 0 \Rightarrow v = 0$ if and only if V is torsionless; and in this case there is a canonical homomorphism of W onto a total submodule of* $\mathrm{Hom}(V, R)$.

Proof. If such a m. c. exists, then for every $w \in W$ consider the homomorphism $\varphi_w : v \to (v, w)$ which is an element of $\mathrm{Hom}_R(V, R)$. Furthermore, since $(v, \varphi_w) = (v, w)$ it follows by the requirement of the theorem is that $\varphi_W = \{\varphi_w \mid w \in W\}$ be clearly a total submodule of $\mathrm{Hom}(V, R)$—hence V is torsionless.

Conversely, if V is torsionless the classical m. c. $M = (R, V, {}^*V, \mathscr{E})$ satisfies the requirement of our theorem by definition of the torsionless property of V.

Theorem 2. *Let $M = (R, V, W. S)$ be a m. c. Then the following are equivalent*:

(1) (a) *V_S faithful*: *i.e.* $V^s = 0 \Rightarrow s = 0$; *and* (b) $[W, v] = 0 \Rightarrow v = 0$

(2) (a) $[W, V]s = 0 \Rightarrow s = 0$; *and* (b) $(V, W)v = 0 \Rightarrow v = 0$.

In either of these cases we have $d({}_RV) = d({}_SS)$.

Proof. If (1) holds and $[W, V]s = 0$ then $[W, Vs] = 0$ and by (b) it follows that $Vs = 0$ and thus (a) implies that $s = 0$. Similarly, if $(V, W)v = 0$ then $V[W, v] = 0$ hence (a) yields that $[W, v] = 0$ and so $v = 0$ by (b), and (2) is proved.

If (2) is valid and $Vs = 0$ then $0 = [W, Vs] = [W, V]s$ which implies $s = 0$ by (a). Also if $[W, v] = 0$ then $0 = V[W, v] = (V, W)v$ and thus $v = 0$ which proves (1).

In either case set $V_1 \oplus \cdots \oplus V_k \subseteq V$ be a direct sum of R-submodules which is in V, then $[W, V_1] \oplus \cdots \oplus [W, V_k]$ is a direct sum of left ideals in S. Indeed $S[W, V_i] = [SW, V_i] \subseteq [W, V_i]$ and $[W, V_i] \neq 0$ by (1b). So $[W, V_i]$ is a non-zero left ideal in S. If $\sum s_i = 0$, $s_i \in [W, V_i]$ then $Vs_i \subseteq V[W, V_i] = (VW)\, V_i \subseteq V_i$. Thus $\sum Vs_i = 0$ implies that $Vs_i = 0$ and hence (1a) yields that $s_i = 0$. This proves that the sum $[W, V_1] + \cdots + [W, V_k]$ is direct and, therefore, $d({}_RV) \leqslant d({}_SS)$.

Now if $L_1 \oplus \cdots \oplus L_t$ is a direct sum of left ideals in S then $VL_1 \oplus \cdots \oplus VL_t$ is a direct sum of R-submodules in V. Indeed, since $L_i \neq 0$, $VL_i \neq 0$ by (1a) and clearly $R(VL_i) = (RV)L_i \subseteq VL_i$ thus VL_i is a non zero left R-submodule of V. If $\sum v_i = 0$ with $v_i \in VL_i$ then $\sum [W, v_i] = 0$ and since

$[W, v_i] \subseteq [W, VL_i] = (W, V)L_i \subseteq RL_i \subseteq L_i$, it follows that $[W, v_i] = 0$ for every i. Hence (1a) implies that $v_i = 0$ i.e. $VL_1 + \cdots + VL_t$ is direct and so $d(_RV) \geqslant d(_SS)$ which completes the proof of the theorem.

By symmetry we obtain the following corollary which will be used later.

COROLLARY 3. *The following are equivalent for a m. c.* $M = (R, V, W, S)$:

(1) (a) $Wr = 0 \Rightarrow r = 0$; *and* (b) $(V, w) = 0 \Rightarrow w = 0$

(2) (a) $(V, W)r = 0 \Rightarrow r = 0$; *and* (b) $[W, V]\, w = 0 \Rightarrow w = 0$.

In either case, $d(_RR) = d(_SW)$.

This corollary is a simple consequence of applying the previous theorem to the set (S, W, V, R) which is also a m. c. with the changing of the role of the maps $(\, , \,)$ and $[\, , \,]$.

In order to extend the theory of primitive rings with a minimal left ideal to rings with a uniform left ideal we extend the notion of density as follows: Let $_SW$ be a lift S-module, and $R \subseteq \mathrm{Hom}_S(W, W)$ then:

DEFINITION. R is dense in $\mathrm{Hom}_S(W, W)$ if for every large submodule W_0 of W, and $T \in \mathrm{Hom}_S(W_0, W)$ and $_SU \subseteq {}_SW$ of finite $d(_SU)$, there exists $a, b \in R$, $U_0 \subseteq U$ such that (1) $d(_SU_0) = d(_SU)$, (2) $U_0a \subseteq U_0$ and a regular on U_0 (i.e. $ua = 0$, $u \in U_0 \Rightarrow u = 0$), and (3) $U_0(aT - b) = 0$.

An equivalent description of the density of R for modules over Öre-domains S is the following:

Let $\bar{S}$ be the ring of quotient of S, $\bar{W}$ be the module of quotients of W, then R is dense in $\mathrm{Hom}_S(W, W)$ if for every $T \in \mathrm{Hom}_{\bar{S}}(\bar{W}, \bar{W})$ and every $U \subseteq W$ of finite dimension there exists $a, b \in R$ such that $u_i(aT - b) = 0$ for some base $\{u_i\}$ of U, a regular on U_0 and, $u_ia \in U_0$.

The equivalence of these two definitions is readily verified, and it will also be obtained later.

The classical density is given by an element a which is the identity on U. With this definition we are able to extend the fundamental theorem on rings with a minimal left ideal [6, Structure theorem, p. 75]:

THEOREM 4. *The following are equivalent properties for a ring* R:

(1) *R is semi-prime; R has a uniform faithful left ideal* $_RL$ *and* $Z(_RR) = 0$.

(2) *There exists a m. c.* $M = (R, V, W, S)$ *such that* $d(_RV) < \infty$ V *is both* R *and* S *faithful, and the product* $(\, , \,)$ *satisfies*:

$$(V - W): \quad (v, w) = 0 \text{ if and only if } v = 0 \text{ or } w = 0;$$
$$(V = W): \quad (V, W)v = 0 \text{ implies } v = 0.$$

(3) *R is a dense ring of linear transformations in an endomorphism ring* $\mathrm{Hom}({}_SW, {}_SW)$ *with S a left Öre domain, ${}_SW$ torsion free and R contain a linear transformation r such that $d({}_SWr) < \infty$.*

Proof. (1) $\Rightarrow$ (2)

Let ${}_RL$ be a left uniform ideal. Consider the standard m. c. $M = (R, L, {}^*L, E)$, ${}^*L = \mathrm{Hom}_R(L, R)$ and $E = \mathrm{Hom}_R(L, L)$. This context will satisfy (2). But first we need the following known result.

LEMMA 5. [8] *Let ${}_RX$ be a uniform module, and $\varphi : {}_RX \to {}_RY$ then either φ is a monomorphism or $X\varphi \subseteq Z({}_RY)$.*

Indeed, if $\mathrm{Ker}\,\varphi \neq 0$ and $\varphi \neq 0$ let $\eta = \xi\varphi \in X\varphi$. For every left ideal ${}_RL$ in R, if $L\xi \cap \mathrm{Ker}\,\varphi = 0$ then $L\xi = 0$ since X is uniform. Hence $L\eta = (L\xi)\varphi = 0$ so that $L \subseteq (0 : \eta)$. If $L\xi \cap \mathrm{Ker}\,\varphi \neq 0$ then there exist $\ell \in L$ such that $0 \neq \ell\xi \in \mathrm{Ker}\,\varphi$ and so $\ell(\xi\varphi) = 0$ i.e. $0 \neq \ell \in (0 : \eta)$ which means that $\eta \in Z({}_RY)$. Q.E.D.

The conclusion of the proof that (1) $\Rightarrow$ (2) is simple: $d_R(L) = 1 < \infty$ and hence $(V - W)$ holds: for if $(v, w) = 0$ $v \in L$ $w \in {}^*L$ then if $v \neq 0$ it follows by the previous lemma that $(V, w) \in Z(R)$. But $Z(R) = 0$ hence $(V, w) = 0$ but $w \in \mathrm{Hom}(L, R)$ and so the definition of homomorphisms implies that $w = 0$. Next $(V = W)$ holds: indeed ${}^*L = W$ includes all maps $\ell \mapsto \ell x$ obtained by right multiplication by an arbitrary element $x \in R$. Hence $(V, W)v = 0$ means in our case, in particular, that $(LR)\ell = 0$. If $\ell \neq 0$ then since $\ell \in L$, ℓ generates a nilpotent left ideal in R, but R is semi prime. Thus $\ell = 0$, and so $(V = W)$ holds.

Finally ${}_RL$ is R-faithful by assumption, and L_S is S-faithful by definition of $S = \mathrm{Hom}_R(L, L)$.

To prove (2) $\Rightarrow$ (3), we show first:

LEMMA 6. *If* (2) *holds, then S is a left öre-domain; furthermore V_S, W_S are torsion free, i.e. $vs = 0$ implies $v = 0$ or $s = 0$ and similarly $sw = 0$ implies $w = 0$ or $s = 0$. Also W_R is faithful and ${}_RV$ is uniform.*

Proof. S is a domain, for let $st = 0$ then $(Vs, tW) = (V, stW) = 0$ hence, if $s \neq 0$, $t \neq 0$ then since V_S, ${}_SW$ are faithful $Vs \neq 0$, $tW \neq 0$ but then $(Vs, tW) = 0$ contradicts condition $(V - W)$ of (2).

Next if $vs = 0$, if $v \neq 0$, then $0 = (vs, W) = (v, sW)$, hence $sW = 0$ by $(V - W)$. But then $0 = (V, sW) = (Vs, W)$ and so $Vs = 0$. Now V_S is faithful, and hence $s = 0$. Similarly if $sw = 0$, then $(Vs, w) = (V, sw) = 0$ and so either $w = 0$ or $Vs = 0$ by $(V - W)$, and then $s = 0$.

W_R is faithful, for let $Wr = 0$ then $0 = (V, Wr)V = (V, W)rV$ and so $rV = 0$ by $(V = W)$. Since ${}_RV$ is faithful it follows that $r = 0$.

We can apply Theorem 2 to our case, since V_S is faithful; and if $[W, v] = 0$ then $V[W, v] = (V, W)v = 0$ and so $v = 0$ by $(V = W)$. Which proves (1) of Theorem 2.

It follows now by Theorem 2 that $d({}_SS) = d({}_RV) < \infty$. But (as it is well known) a domain of finite dimension is an Öre domain. Indeed, if a, b are not zero and $Sa \cap Sb = 0$ then $Sa + Sab + \cdots + Sab^n$ is a direct sum, since if $\sum_{i=m}^{n} x_i ab^i = 0$ and $x_m \neq 0$ then $(x_m a + \sum x_i ab^{i-m})\, b^m = 0$ and therefore $x_m a + \sum_{i>m} x_i ab^{i-m} = 0$ and so $0 \neq x_m a = -\sum x_i ab^{i-m} \in Sa \cap Sb = 0$ which is a contradiction, hence $Sa \cap Sb \neq 0$ i.e. $d({}_SS) = 1$.

Applying again Theorem 2 and we have $d({}_RV) = d({}_SS) = 1$. i.e., ${}_RV$ is uniform, and the proof of Lemma 6 is complemented.

We shall need also the following:

LEMMA 7. *If $M = (R, V, W, S)$ is a m. c. with an S—an Öre domain, and the product $[\ ,\ ]$ satisfies: "$[w, V] = 0$ implies $w = 0$" and let $w_1, \ldots, w_n$ be S-independent elements in V then there exists $v_1, \ldots, v_n \in V$ and $\bar{w}_1, \ldots, \bar{w}_n$ in $W_0 = Sw_1 + \cdots + Sw_n$ such that $[\bar{w}_i, v_k] = s\delta_{ik}$ for some regular $s \in S$.*

Proof. Consider the division ring of quotients $\bar{S}$ of S and $\overline{W}$ the module of quotients of W and $\overline{W}_0 = \sum_{i=1}^{n} \bar{S}w_i$. The elements of R acts also on $\overline{W}$ by setting $(q^{-1}w)r = q^{-1}(wr)$. The elements $r \in (V, W_0)$ will map W_0 into itself as $W_0(V, W_0) = [W_0, V]\, W_0 \subseteq W_0$. Hence, these elements will map $\overline{W}_0$ into itself. Let $r \in (V, W_0)$ such that $\overline{W}_0 r$ is of maximum S-dimension, then r is regular on $\overline{W}_0$. If it were not, then $\mathrm{Ker}(r) \neq 0$ and $\dim(\overline{W}_0 r) < \dim(\overline{W}_0)$ and therefore, one of the w_i, say w_n, is $\bar{S}$ independent of the elements of $\overline{W}_0 r$, and let $0 \neq w_0 \in \mathrm{Ker}(r)$ and then exists v_0 such that $[w_0, v_0] \neq 0$. Consider the element $r' = r + (v_0, w_n) \in (V, W_0)$ and if $wr' = 0$ then $0 = wr + w(v_0, w_n) = wr + [w, v_0]\, w_n$ and hence $wr = 0$ and $[w, v_0]\, w_n = 0$. Thus $\mathrm{Ker}(r') \subseteq \mathrm{Ker}(r)$ and since for $w_0 \in \mathrm{Ker}(r)$, $w_0 r' = w_0 r + [w_0, v_0]\, w_n$ and $[w_0, v_0]\, w_n \neq 0$, it follows that $\mathrm{Ker}(r') \neq \mathrm{Ker}(r)$. This implies that $\dim(\overline{W}_0 r') > \dim(\overline{W}_0 r)$ which contradicts the maximality of $\dim(\overline{W}_0 r)$. Thus r is regular and $\overline{W}_0 r = \overline{W}_0 = \sum_{i=1}^{n} \bar{S}w_i$. $r \in (V, W_0)$ and so $r = \sum_{i=1}^{n} (v_i, w_i)$, and since $w_j \in \overline{W}_0 r$, there exists w_j' such that $w_j' r = w_j$. By choosing a common denominator we may assume $w_j' = s^{-1}\bar{w}_j$ and $w_j \in W_0$, then $\bar{w}_j r = sw_j$ which yields $sw_j = \bar{w}_j r = \sum_i \bar{w}_j(v_i, w_i) = \sum_{i=1}^{n} [\bar{w}_j, v_i]\, w_i$ and therefore, $[\bar{w}_j, v_i] = s\delta_{ij}$. Q.E.D.

COROLLARY 8. *The set $\bar{w}_1, \ldots, \bar{w}_n$ are left S-independent, and the element $\bar{r} = \sum (v_i, \bar{w}_i)$ is regular on W_0 and satisfies $\bar{w}_i \bar{r} = s\bar{w}_i$. In particular, if S is a division ring we can choose $s = 1$.*

Consider, $\bar{w}_j \bar{r} = \sum_i \bar{w}_j(v_i, \bar{w}_i) = \sum [\bar{w}_j, v_i]\, \bar{w}_i = s\bar{w}_j$ and if $\{\bar{w}_j\}$ will be shown to be S-independent, then $\bar{r}$ is regular on $\overline{W}_0$. Indeed, $\sum s_j\bar{w}_j = 0$ implies $\sum [s_i\bar{w}_j, v_i] = s_j s = 0$ and hence $s_j = 0$.

We are now in position to prove: (2) $\Rightarrow$ (3).

If (2) holds, then by Lemma 6, S is an Öre domain; furthermore Lemma 7 is applicable. For, if $[w, V] = 0$ then $0 = V[w, V] = (V, w)V = 0$ and as $_RV$ is faithful it follows that $(V, w) = 0$ and thus $w = 0$. Since W_R is faithful, there is a natural embedding $R \to \mathrm{Hom}_S(W, W)$. To prove that this embedding is dense let $_SW^0$ be large in $_SW$, $_SW_0 \subseteq W^0$ and $d(_SW_0) = n < \infty$. Choose, by Lemma 7, the elements $\bar{w}_i \in W_0$ such that $[\bar{w}_i, v_k] = s\delta_{ik}$, and let $T \in \mathrm{Hom}_S(W^0, W)$, and put $a = \sum (v_i, \bar{w}_i)$, $b = \sum (v_i, \bar{w}_iT)$ then for every $w \in W$

$$wb = \sum w(v_i, \bar{w}_iT) = \sum [w, v_i](\bar{w}_iT) = \left(\sum [w_iv_i]\, \bar{w}_i\right) T$$
$$= \left(\sum w(v_i, \bar{w}_i)\right) T = waT.$$

Thus $W(b - aT) = 0$ and so $b = aT$ with a regular on W_0 by Corollary 8.

R contains a linear transformation of finite dimension, for choose $r = (v, w) \neq 0$ then $Wr = [W, v]w$ is of dimension 1.

Remark. Note in particular that for $w = \bar{w}_j$, we have $\bar{w}_jaT = s\bar{w}_jT = \bar{w}_jb$ and if S is a division ring, where we may choose $s = 1$, then the classical density follows, since in this case, $\bar{w}_jT = \bar{w}_jb$.

Note that our proof yields a stronger result then the density defined above; namely,

COROLLARY 9A. *If $M = (R, V, W, S)$ satisfies (2) of Theorem 4, then for every $T \in \mathrm{Hom}_{\bar{S}}(\overline{W}, \overline{W})$, and $U_0 \subseteq W$ of dimension n—there exists $a, b \in R$ such that $b = aT$ and $w_ia = sw_i$ for some base $\{w_i\}$ of U_0 and $s \neq 0$ in S.*

The proof that (3) $\Rightarrow$ (1).

First observe that for W over Öre-domains S the equivalence of the preceding definitions of density follows from the fact that large submodules W_0 of W are those which contain a base of the module of quotient $\overline{W}$ over the division ring of quotient $\bar{S}$, and hence the homomorphisms $T \in \mathrm{Hom}(W_0, W)$ are completely determined by their extensions in $\mathrm{Hom}_{\bar{S}}(\overline{W}, \overline{W})$ defined by the effect of T on the base of $\overline{W}$ contained in W_0.

We shall, therefore, use only the second definition of the density and first we prove

LEMMA 9. *If (3) holds then for arbitrary $w_0 \neq 0$, $w \neq 0$ in W, there exists $\rho \in R$ such that $Wr_0 \subseteq Sw$ and $w_0r_0 = sw \neq 0$.*

Proof. Let $r \in R$ such that $d(_SWr) < \infty$, and let $w_1r,..., w_mr$ be a base of Wr (i.e. of $\overline{Wr}$). First choose $T_1 \in \mathrm{Hom}_S(\overline{W}, \overline{W})$ such that $w_0T = w_1$ and zero on a complementary base of w_0 then by the density property let $w_0(a_1T_1 - b_1) = 0$ and $0 \neq w_0a_1 \in Sw_0$. Then choose $T_2 \in \mathrm{Hom}(\overline{W}, \overline{W})$ such that $(w_1r)\, T_2 = w$, $(w_ir)\, T_2 = 0$ for $i > 1$ and T_2 let annihilate a complementary base of the $\{w_ir\}$, and let $(w_ir)(a_2T_2 - b_2) = 0$, where a_2, b_2 were chosen by the density property, so that $0 \neq (w_1r)\, a_2 \in S(w_1r)$, since otherwise $Wra_2T_2 = 0$, but a_2 is regular on Wr.

The required element is then $r_0 = b_1rb_2$. Indeed, $Wr_0 \subseteq Wrb_2 \subseteq Wr(a_2T_2) \subseteq Sw$; and

$$\begin{aligned} w_0b_1rb_2 = w_0(a_1T_1)\, rb_2 = s_1w_0T_1rb_2 = s_1w_1rb_2 = s_1w_1r(a_2T_2) &= s_1s_2(w_1r)\, T_2 \\ &= sw \neq 0. \end{aligned}$$

In particular, it follows that if (3) holds then there exists $r \in R$ such that $d(_SWr) = 1$. Let $L = \{x \in R;\, Wx \subseteq Wr\}$ then $L \neq 0$ is the uniform ideal satisfying (1):

L is uniform, for let $x, y \in L$ be non zero, then since $d(Wx) = d(Wy) = d(Wr) = 1$ both Wx and Wy are large in Wr and $Wx \cap Wy \neq 0$ so let $0 \neq wy = \bar{w}x$. In the quotient module $\overline{W}$, the image $\overline{Wy}$ is one dimensional, hence $\ker(y)$ is of codimension 1, and we can find a basis $\{w_i \in W\}$ of $\overline{W}$ such that $w_1 = w$, and $w_i \in \mathrm{Ker}(y)$ for $i > 1$. Consider $T \in \mathrm{Hom}_S(\overline{W}, \overline{W})$ given by $w_1T = \bar{w}$ and $w_iT = 0$ for $i > 1$, then $w_1(Tx - y) = \bar{w}x - wy = 0$ by definition of $\bar{w}, w = w_1$, and also $w_i(Tx - y) = 0$ for $i > 1$; hence $Tx - y = 0$. On the other hand by the density property we have $w_1(aT - b) = 0$ for some $a, b \in R$, $0 \neq w_1a \in Sw_1$. Let $r_0 \in R$ be chosen by Lemma 9 such that $Wr_0 \subseteq Sw_1$ and $w_1r_0 \neq 0$, then $Wr_0(aT - b) \subseteq Sw_1(aT - b) = 0$. Hence $W(r_0aT - r_0b) = 0$ and note that $w_1r_0a \neq 0$. Consequently, $r_0aT - r_0b = 0$. This implies that $0 = (r_0aT - r_0b)x = (r_0a)y - (r_0b)x$, which means that $r_0ax = r_0by \in Rx \cap Ry$ and $r_0ax \neq 0$ since $w_1r_0ax \neq 0$. This completes the proof that $_RL$ is uniform.

$_RL$ is faithful, and in fact we prove more: if $wL = 0$ then $w = 0$. This will imply that if $rL = 0$ then $WrL = 0$ and, therefore, $Wr = 0$, but as W_R is faithful, since R is a subring of $\mathrm{Hom}_S(W, W)$—it follows that $r = 0$. Indeed, let $wL = 0$ and let $0 \neq w_0r \in Wr$ for some $r \in L$. If $w \neq 0$, choose $b \in R$, by Lemma 9, so that $wb = sw_0$, and then $0 \neq sw_0r = w(br) = 0$ since $br \in L$, which is a contradiction. Thus $w = 0$. Q.E.D.

$Z(_RR) = 0$: If it were not, then let $0 \neq z \in Z(_RR)$, and $w_0 = wz \neq 0$, for some $w \in W$. By Lemma 9, it follows that the left ideal $L_0 = \{r \in R;\, Wr \subseteq Sw\}$ is non zero, and since $z \in Z(R)$ we have $0 \neq r_0 \in L_0 \cap (0 : z)$, i.e. $r_0z = 0$. Now, for some $w_1 \in W$, $w_1r_0 = sw \neq 0$ since W_R is faithful; hence, $0 \neq sw_0 = swz = w_1(r_0z) = 0$ which is a contradiction.

Remark 10A. We have used the semi-primeness of R only in the part which shows that (1) $\Rightarrow$ (2), where we had to show $(V = W)$, which was essential in permitting the application of Theorem 2. We conjecture that the requirement of semi-primeness can be dropped, though we are not able to prove it, we list a few conditions which may replace semi-primeness in the statement of (1) and apparently seems less restrictive:

(1) The two sided ideal LR has no non zero right annihilator in L.

(2) L has no non zero nilpotent left ideals of R.

(3) ${}_R(LR)$ is large, as a left ideal, in R.

Each of these conditions implies $(V = W)$; indeed, as in the original proof of $(V = W)$, we observed that W contains all left multiplications by elements of R, hence $(V, W)v = 0$ means that $(LR)\ell = 0$ for $\ell\,(=v)$ in ${}_RL$. If (1) holds then $\ell = 0$, i.e. $(V = W)$ is valid. If (2) holds then, the left ideal generated by ℓ in R is nilpotent and hence it is zero so that $\ell = 0$. If (3) holds then $(0 : \ell) \supseteq LR$ is large in R and thus $\ell \in Z({}_RR) = 0$. Q.E.D.

To complete the proof of Theorem 4, we wish to show that (3) $\Rightarrow$ any of conditions of Remark 10A as well as that R is semi-prime, which of course will show that under the conditions of (1) Theorem 4, all these are equivalent:

R is semi-prime, for let $aRa = 0$; if $a \neq 0$, let $wa \neq 0$ and by the density it follows that there exist r such that $(wa)r = sw \neq 0$ and so $0 = (wa)\,ra = swa \neq 0$, which is a contradiction. Semi primeness of R, clearly implies (1), since if $(LR)\ell = 0$, then $\ell R\ell = 0$ and so $\ell = 0$. Also implies (2), as R as well as L has no non-zero nilpotent left R-ideals. Condition (3) also follows, since if $r \neq 0$ then $rL \neq 0$ let $\ell_0 = r\ell \neq 0$ and then $(LR)\,\ell_0 \neq 0$ by (1), which implies $(LR)r \neq 0$. Thus $Rr \cap (LR) \supseteq (LR)r \neq 0$, i.e. LR is large. Q.E.D. This completes the proof of Theorem 4 and Remark 10A.

Rings which satisfy Theorem 4 satisfy the following properties:

THEOREM 10B. *If R satisfies one of the conditions of Theorem* 4, *then*:

(1) *R is prime.*

(2) *For every finite dimensional submodule ${}_SW_0 \subseteq W$ there exists $r \in R$ and $w_1, \ldots, w_n \in W_0$ a maximal set of S-independent elements in W_0, and such that $w_ir = sw_i$ for some $s \neq 0$ in S.*

(3) *For every ideal A in R, $0 \neq A \cap (V, W)$ and it is large in A. In particular (V, W) is large in R (compare with* [14]*).*

Proof. Let $xRy = 0$, then $(xV, Wy) = x(V, W)y = 0$. It follows, therefore, by (2) of Theorem 4 that either $xV = 0$ or $Wy = 0$, and the same result implies that either $x = 0$ as ${}_RV$ is faithful, or $y = 0$ by Lemma 6 and this proves (1).

Condition (2) is a restatement of Lemma 7 and Corollary 8 in our case.

If A is an ideal in R, then for every $0 \neq a \in A$, considering $a \in \mathrm{Hom}_S(W, W)$ we choose $(v, wa) = (v, w)a \in Ra \cap (V, W) = Ra \cap (A \cap (V, W))$ since $a \in A$. As R is prime, one can always determine v, w so that $(v, w)a \neq 0$, otherwise $(V, W)a = 0$; thus $(V, W)a \neq 0$. This proves (3).

Further properties of these rings are:

THEOREM 10C. *If R satisfies any of the conditions of Theorem* 4 *then*:

(1) *The complete left ring of quotients of R is* $\mathrm{Hom}_{\bar{S}}(\bar{W}, \bar{W})$, *where $\bar{S}$ is the Öre left ring of quotients of S, and $\bar{W}$ the module of quotients of W.*

(2) *For every $W_0 \subseteq W$ of finite S-dimension m, there exist $R_0 \subseteq R$, a left ideal in R and a homomorphism $\psi : R_0 \to \mathrm{Hom}_S(W_0, W_0)$ such that the left (Öre) ring of quotients of $R_0/\mathrm{Ker}\,\psi$ is isomorphic with $\bar{S}_m \simeq \mathrm{Hom}_{\bar{S}}(\bar{W}_0, \bar{W}_0)$. Furthermore $R_0/\mathrm{Ker}\,\psi \supseteq (S_1)_m$ where S_1 is an Öre domain $\subseteq S$ which have the same ring of quotients $\bar{S}$.*

(3) *In particular, if $n = d({}_SW) < \infty$, R has an Öre—left ring of quotient $\simeq \bar{S}_n$, and $R \supseteq (S_0)_n$, where S_0 is large in S and have the same left ring of quotients* (*Faith-Utumi, e.g.* [5]).

(4) *If $d({}_RR) < \infty$ then* (3) *holds and $d({}_SW) = d({}_RR)$.*

(5) *Generally $d(R_R) = d(V_S) \geqslant d({}_SW) \geqslant d({}_RR)$.*

(6) *If $d(R_R) < \infty$ and R satisfies* (1) *of Theorem* 4 *$d(R_R) = d({}_RR)$ and both right and left ring or quotients are canonically isomorphic.*

Proof. To prove (1) it suffices to show that the natural embedding of R into $\mathrm{Hom}_S(W, W)$ into $\mathrm{Hom}_{\bar{S}}(\bar{W}, \bar{W}) = \mathscr{H}$ is a monomorphism and R is large in ${}_R\mathscr{H}$. Since then $\mathscr{H}$ is its own ring of quotient and so $\mathscr{H}$ is also the quotient ring of R [11]; and indeed, if $0 \neq T \in \mathscr{H}$ and $wT \neq 0$, $w \in W$ then $swT \in W$ for some $s \in S$ and so $0 \neq (V, swT) = (V, sw)T \subseteq RT \cap R$ as required, note that $(v, wT) = (v, w)T$ as elements in $\mathrm{Hom}(\bar{W}, \bar{W})$. Clearly the definition $(s^{-1}w)r = s^{-1}(wr)$ turns $\bar{W}$ into a faithful R-module and this yields the injection of R into $\mathscr{H}$.

To prove (2): Let ${}_SW_0$ be of dimension m and set $R_0 = \{r \in k;\ W_0 r \subseteq W_0\}$. Clearly $R_0 \supseteq (V, W_0)$, since $W_0R_0 = W_0(V, W_0) = [W_0, V]\,W_0 \subseteq W_0$. Multiplication by $r_0 \in R_0$ yield an S-homomorphism of W_0, hence the mapping $\psi : R_0 \to \mathrm{Hom}_S(W_0, W_0)$ given by $(w)\,r_0^{\psi} = wr_0$, $w \in W$ is a well defined homomorphism, and $\ker \psi = \{t \in R_0 ;\ W_0 t = 0\}$. Thus ψ induces a monomorphism of $R_0/\ker \psi \to \mathrm{Hom}_S(W_0, W_0)$.

Let $T \in \mathrm{Hom}_{\bar{S}}(\bar{W}_0, \bar{W}_0)$. Choose $w_1, ..., w_m$ in W_0 which are S-independent and for which we have $v_i \in V$ with $[w_i, v_k] = s\delta_{ik}$ $0 \neq s \in S$. The w_i are also a base of $\bar{W}_0$; $w_iT \in \bar{W}_0$ which is of finite dimension, hence we can find $q \neq 0$, $q \in S$ so that $(qw_i)T \in W_0$. By replacing qw_i by w_i, the orthogonality property for $[w_i, v_k]$ will hold for a different non zero s, and for such w_i both w_i, $w_iT \in W_0$. We can then extend T to an element, to be denoted also by T, of $\mathrm{Hom}(W_1, W)$ where W_1 is the S-large submodule of S generated by these w_i and a complementary base, and where we set $w_jT = 0$ if w_j in the complementary base (i.e. $j > m$). Then put $a = \sum (v_i, w_i)$ $b = \sum (v_i, w_iT)$ and we have $b - aT = 0$. Both $a, b \in R_0$ and a is regular on W_0 which proves that every $T \in \mathrm{Hom}(\overline{W}_0, \overline{W}_0)$ can be written in the form $a^{-1}b$, with a regular in $R_0/\mathrm{Ker}\,\psi$, and this proves the first part of (2).

To prove the second part of (2): let w_i, v_i as above i.e. $[w_i, v_k] = s\delta_{ik}$ $0 \neq s \in S$ and $W_0 = \sum Sw_i$. Consider the ring $sS = S_1$ and the module $W_1 = \sum S_1 w_i$. One readily verifies that S_1 and S have the same left ring of quotients. Consider $(S_1)_m$ as a subring of $\mathrm{Hom}_{S_1}(W_1, W_1)$, by noting that W_1 is a free S_1-module, and let $T_1 \in (S_1)_m$, then $w_iT_1 = s\bar{w}_i$ for $\bar{w}_i \in W_0$. Put $a = \sum (v_i, \bar{w}_i)$ then $a \in R_0$ since $W_0 a \subseteq \sum [W_0, v_i]\bar{w}_i \subseteq W_0$ and also:

$$w_k a = \sum w_k(v_i, \bar{w}_i) = \sum [w_k, v_i]\, \bar{w}_i = s\bar{w}_i = w_k T_1$$

Thus $wa = wT_1$ for every $w \in W_1$.

Let R_1 be the subring of all $a \in R_0$ for which there exists a $T_1 \in \mathrm{Hom}(W_1, W_1)$ and $wa = wT_1$ for every $w \in W_1$. First, note that by taking $T_1 = 0$, the set $R_1 \supseteq \mathrm{Ker}\,\psi$ since then $wa = 0$ for $w \in W_1$ and hence also for $w \in W_0$. Consider the map: $a \mapsto T_1$ where $wa = wT_1$, then if we consider the unique extension $\bar{T}_1$ of T_1 in $\mathrm{Hom}_S(W_0, W_0)$, then the map $a \to \bar{T}_1$ is the ψ defined above. From this one readily verifies that R, is a ring, and ψ induces an isomorphism of $R_1/\mathrm{Ker}\,\psi$ with a ring containing $(S_1)_m$.

Q.E.D.

Assertion (3) of our theorem is now a special case of (2), for $W = W_0$.

If (4) holds i.e. $d({}_RR) = n < \infty$, then Corollary 3 is applicable. Indeed, W_R is faithful by Lemma 6, and (1b) of Corollary 3 is a special case of $(V - W)$. Thus $d({}_RR) = d({}_SW) = n < \infty$ and one applies (3).

To prove (4), consider the m.c. $M^0 = (R^0, W^0, V^0, S^0)$ where R^0, S^0 are the opposite rings of R and S respectively, W^0, V^0 are the opposite modules of W, V respectively, and the bimodule structure is given by $r^0w^0 = (wr)^0$, $w^0s^0 = (sw)^0$ for $w \in W$, $r \in R$, $s \in S$ and a similar definition for V^0 as an $S^0 - R^0$ bimodule. The product $(\,,\,) : W^0 \times V^0 \to R^0$ is given by $(w^0, v^0) = (v, w)^0$ and similarly $[v^0, w^0] = [w, v]^0$. It is easy to verify that M^0 is also a m.c.

M^0 will satisfy the conditions of Corollary 3. Indeed, V^0 is R^0 faithful since V is R-faithful, and if $(W^0, v^0) = 0$ then $(v, W) = 0$ and hence $v = 0$ by $(V - W)$. Hence it follows by this corollary that $d({}_{R^0}R^0) = d({}_{S^0}V^0)$; but $d({}_{R^0}R^0) = d(R_R)$, $d({}_{S^0}V^0) = d(V_S)$ hence $d(R_R) = d(V_S)$.

Finally $d(V_S) \geqslant d({}_SW)$: Indeed, if $d({}_SW) \geqslant n$, then there exists, by

Lemma 7, w_i, v_i, $1 \leqslant i \leqslant n$ such that $[w_i, v_k] = s\delta_{ik}$, $s \neq 0$. But then $\{v_k\}$ are S-independent, for if $\sum v_k s_k = 0$ then $0 = [w_i, \sum v_k s_k] = [w_k, v_k]\, s_k = ss_k = 0$ implies $s_k = 0$. Thus V_S has also n elements which are S-independent. Consequently $d(V_S) \geqslant d({}_SW)$.

Now if $m = d(R_R) < \infty$, then $d({}_RR) < \infty$ and therefore $d({}_SW)$ and $d(V_S)$ are finite. By (3) it follows that R has a left Öre-ring of quotients isomorphic with $\bar{S}_n \cong \mathrm{Hom}_{\bar{S}}(\bar{W}, \bar{W})$, where $\bar{S}$ the left ring of quotient of S, and $\bar{W}$ the left module of quotient of W.

Consider again the m.c. $M^0 = (R^0, W^0, V^0, S^0)$. This will also satisfy conditions (2) of Theorem 4. Clearly W^0 is R^0-faithful since W is R-faithful, by Lemma 6. W^0 is also S^0-faithful, for if $W^0 s^0 = 0$ then $sW = 0$ and so $0 = (V, sW) = (Vs, W) = 0$ and thus $Vs = 0$, but V_S is faithful so $s = 0$ and also $s^0 = 0$. Next $(V - W)$ clearly holds in our case. Note also that $(V - W)$ implies that for a fixed $v \neq 0$, the mapping $w \mapsto (v, w)$ induces an isomorphism between W_R and the right ideal (v, W) in R. Hence $d(W_R) = d((v, W)_R) \leqslant d(R_R) < \infty$. (Note that this will imply $d(W_R) = 1$). Thus $d({}_{R^0}W^0) = d(W_R) < \infty$.

Condition $(V = W)$ also holds in our case; for let $(W^0, V^0)\, w^0 = 0$, so that $w(V, W) = 0$ but then $[w, V]W = 0$ and this implies that $[w, V] = 0$. Hence $(V, w)V = V[w, V] = 0$, and then $(V, w) = 0$ so that $w = 0$. Q.E.D.

Applying now the previous results to M^0, we obtain that S^0 is a left-Öre-domain and hence S is also a right-Öre-domain. Similarly, R^0 has a left (Öre) ring of quotient which is isomorphic with a matrix ring over a division ring, and hence R has also has a right (Öre) ring of quotients and since then regular elements are both right and left it is known (and easily proved) that both ring of quotients are the same. Q.E.D.

III. Uniqueness

Let ${}_RV$ be a uniform left R-module. The extended contralizer of V is defined as follows [7]:

Consider the union of all $\mathrm{Hom}_R(V_0, V)$ where V_0 ranges over all large submodules of V, and define an equivalence relation in this set by putting $\alpha \equiv \beta$ if $\alpha = \beta$ on some non zero submodule contained in the domain of definitions of both α and β. The equivalence relation together with proper definition of addition and multiplication on the submodules of V—where the product and sum can be defined turn this set into a ring $\mathscr{E} = \mathscr{E}(V)$ known as the centralizer of V.

The uniqueness properties of the uniform modules and their ring of endomorphisms are summarized in the following theorem:

THEOREM 11. *Let R satisfy Theorem 4 and $M = (R, V, W, S)$ be any arbitrary m.c. satisfying condition* (2) *of that theorem. Let ${}_R U$ be any R-faithful and uniform and $Z_R(U) = 0$, then*

(1) *The extended centralizer $\mathscr{E}(U)$ of U is the quotient ring of some fixed subring* $\mathrm{Hom}(U_0, U_0)$ *for a given ${}_R U_0 \subseteq U$. The ring $\mathscr{E}(U)$ is isomorphic with the ring of quotient of S, where S is any ring appearing in any m. c. satisfying Theorem* 4.

(2) *If $M_0 = (R, V_0, W_0, S_0)$ is another m. c. satisfying condition* (2) *of Theorem* 4, *then V is isomorphic with a submodule of V_0, V_0 is isomorphic with a submodule of V, and the rings S, S_0 are isomorphic with equivalent orders of the same division ring $\mathscr{E}(U)$.*

Before proceeding with the proof we need some lemmas.

The first is a well known properties of large submodules.

LEMMA 12. *Let ${}_R X_0$ be a large submodule of ${}_R X$, and $\varphi : X_0 \to Y$ and Y_0 be a R-large submodule of Y, then $Y_0\varphi^{-1}$ is large in X.*

If ${}_R X_1 \subseteq X$ and $X_1 \cap Y_0\varphi^{-1} = 0$ then $(X_1 \cap X_0)\varphi$ is defined and $(X_1 \cap X_0)\varphi \cap Y_0 = 0$. Hence $(X_1 \cap X_0)\varphi = 0$ which means that $X_0 \cap X_1 \subseteq \mathrm{Ker}\,\varphi$. But $0 = X_1 \cap \mathrm{Ker}\,\varphi \subseteq X_1 \cap Y_0\varphi^{-1}$, hence, $X_0 \cap X_1 = 0$ and as X_0 is large this proves that $X_1 = 0$. Q.E.D.

LEMMA 13. *Let ${}_R V$ be a faithful R-module, $Z({}_R V) = 0$ and ${}_R L$ a uniform left ideal then V contains a submodule isomorphic with L.*

Indeed, $LV \neq 0$, so let $v \in V$ be such that $Lv \neq 0$. The homomorphism $L \to Lv$ defined by $\ell \mapsto \ell v$ is an isomorphism by Lemma 5. Thus, $V \supseteq Lv \simeq L$.

Proof of Theorem 11. First we prove: that if U_0 is a non zero submodule of U then $\mathscr{E}(U_0)$ and $\mathscr{E}(U)$ are canonically isomorphic, and hence we can identify them, and then both $\mathrm{Hom}(U_0, U_0)$ and $\mathrm{Hom}(U, U)$ become subrings of $\mathscr{E}(U) = \mathscr{E}(U_0)$:

An element in $\mathscr{E}(U)$ is represented by a homomorphism α defined on some submodule of U, and as we can replace the domain of definition of α by $U_0 \cap U_0\alpha^{-1}$ which is non zero, and in fact large in U by Lemma 12, it follows that the class of α is represented by elements of a class of $\mathscr{E}(U_0)$. Conversely, if γ represents a class of $\mathscr{E}(U_0)$ then clearly γ is defined on a submodule of U and, therefore, it represents a class of $\mathscr{E}(U)$, which will depend only on the class represented by γ. This yields, readily, an identification between $\mathscr{E}(U)$

and $\mathscr{E}(U_0)$. The embedding of $\mathrm{Hom}(U, U)$ and similarly of $\mathrm{Hom}(U_0, U_0)$ is by corresponding to each $\alpha \in \mathrm{Hom}(U, U)$ its class in $\mathscr{E}(U)$. If this class is zero then $\alpha = 0$ on some submodule of U, i.e. $\mathrm{Ker}(\alpha) \neq 0$ but since ${}_RU$ is uniform and $Z({}_RU) = 0$, it follows by Lemma 5, that $\alpha = 0$ in $\mathrm{Hom}(U, U)$. Thus we may consider $\mathrm{Hom}(U, U)$ as well as $\mathrm{Hom}(U_0, U_0)$ as subrings of $\mathscr{E}(U)$.

To prove the first part of Theorem 11, we choose ${}_RU_0 = Lu$ (by Lemma 13) where ${}_RL$ is the uniform ideal given in (1) of Theorem 4. Namely, we prove that $\mathrm{Hom}(U_0, U_0)$ has $\mathscr{E}(U) = \mathscr{E}(U_0)$ as a left ring of quotients. Indeed, if $\alpha \in \mathrm{Hom}(U_0, U_0)$, $\alpha \neq 0$ then α is regular and so there exists $\alpha^{-1} : U_0\alpha \to U_0$, and the class of α^{-1} will be the inverse of the class of α in $\mathscr{E}(U)$. If $\bar{\lambda} \in \mathscr{E}(U_0)$ then it is determined by a representative λ defined on $U_1 \subseteq U_0$ and we can choose $u_1 \in U_1$ such that $Lu_1 \neq 0$. Indeed, let $0 \neq u$ be arbitrary in U_1, then $(LR)L \neq 0$, as we have seen in the proof of Theorem 4 (condition ($V = W$) of that theorem). This will imply that $(LR)Lu \neq 0$ since the $L \to U_0$ given by $\ell \mapsto \ell u$ is an isomorphism. Choose $u_1 \in RLu \subseteq U_1$ such that $Lu_1 \neq 0$. Consider now the map $\alpha : U_0 \to U_0$ which is the composite: $U_0 \to L \to U_1 \to U_0$, where the first map is the inverse of $L \to U_0$ given by $\bar{u} : \ell \mapsto \ell u$, the second map: $L \to U_1$ is given by multiplying by u_1, and $U_1 \to U_0$ is the injection of U_1 as a submodule of U_0 and note that $\alpha \neq 0$. Thus $\alpha \in \mathrm{Hom}(U_0, U_0)$, and in $\mathscr{E}(V)$ the class $\bar{\alpha}\bar{\lambda} \in \mathrm{Hom}(U_0, U_0)$ as $\alpha\lambda$ is represented by a map $\beta : U_0 \to U_0$. Thus $\bar{\lambda} = \bar{\alpha}^{-1}\bar{\beta}$ in $\mathscr{E}(V)$. This concludes the fact that $\mathscr{E}(U) = \mathscr{E}(U_0)$ is the ring of quotient of $\mathrm{Hom}(U_0, U_0)$.

Furthermore, since $U_0 \simeq L$ for a fixed ${}_RL$ chosen independent of U, it follows that all $\mathscr{E}(U)$ are isomorphic with $\mathscr{E}(L)$ since, clearly the latter is isomorphic with $\mathscr{E}(U_0)$.

To prove (2), let M, M_0 be two m. c. of Theorem 4. V and V_0 are then isomorphic to two left ideals satisfying (1) of Theorem 4.

For choose $0 \neq w \in W$, then $(V, w) \simeq V$ by corresponding $v \mapsto (v, w)$ and (V, w) is a left uniform ideal of R. Similarly $V_0 \simeq (V_0, w_0)$. It follows now by Lemma 13 that V_0 contains a submodule isomorphic with V, and V contain a submodule isomorphic with V_0.

We already know by the first part of the proof that $\mathscr{E}(V_0)$ is isomorphic with the ring of quotient of $\mathrm{Hom}_R(V, V) = H(V)$, and $\mathscr{E}(V)$ isomorphic with the ring of quotient of $\mathrm{Hom}_R(V_0, V_0) = H(V_0)$, and both are isomorphic with a fixed $\mathscr{E}(U)$. But we can say more:

There is a homomorphism $\xi : V_0 \to V$, so that $V_0\xi \subseteq V$, and similarly $V\eta \subseteq V_0$. In order to be able to handle both $H(V)$ and $H(V_0)$ as subring of the same ring, we can identify V_0 with a submodule of V, since the ξ given above must be a monomorphism by Lemma 5. If this is the case then both ξ, η determine classes of $\mathscr{E}(V)$ and in fact $\xi = 1$ in $\mathscr{E}(V)$.

For every $h \in H(V)$, $V_0\xi h\eta \subseteq Vh\eta \subseteq V\eta \subseteq V_0$. So $\xi h\eta$ determines an element of $\mathscr{E}(V_0)$. Thus $\xi H(V)\eta \subseteq H(V_0)$ and similarly $\eta H(V_0)\xi \subseteq H(V)$ which means that $H(V)$ and $H(V_0)$ are equivalent orders in $\mathscr{E}(V)$.

Finally we show that $H(V) \supseteq S \supseteq H(V)s$ for some $0 \neq s \in S$, and similarly, $H(V_0) \supseteq S_0 \supseteq H(V_0)s_0$. Indeed, $S \subseteq H(V)$ since V_S is faithful and hence multiplication by elements of S generates unique elements of $H(V)$, in view of which we may assume $H(V) \supseteq S$. Also if $\alpha \in H(V)$, choose $s = [\bar{w}, \bar{v}] \neq 0$, then $(vs)\alpha = (v[\bar{w}, \bar{v}])\alpha = ((v, \bar{w})\bar{v})\alpha = (v, \bar{w})(\bar{w}\alpha) = v[\bar{w}, \bar{v}\alpha]$ and so $s\alpha \in S$. Q.E.D.

This proves first that S and $H(V)$ have the same ring of quotients, and so S and S_0 have the same ring of quotient (after the right changes in V, V_0) and also $(\eta s_0)\ S_0(\xi s) \subseteq S$, $(\xi s)\ S(\eta_0 s_0) \subseteq S_0$ and the two are equivalent orders. Q.E.D.

IV. Applications

(a) A simple case where Theorem 4 holds is a primitive ring R with a minimal left ideal L, then $L = Re$ where is a primitive idempotent e. The ring R satisfies (2) of Theorem 4 with the m. c. $M = (R, Re, eR, eRe)$ where inner and outer product are the ordinary multiplication in R. In this case, $S = eRe$ is a division ring and, therefore, it is its own ring of quotient. Furthermore, the density in this case is the classical density, and we have the classical result [6, Ch. IV].

(b) Let R be a prime which contains an element $a \in R$ such that aRa is a ring without zero divisors then the m. c. (R, Ra, aR, aRa), with the products ( ), [ ], the multiplication in R satisfies conditions (2) of Theorem 4 (with the possible exception that $d({}_R Ra) < \infty$):

Indeed, Ra is R-faithful, since $xRa = 0$ implies $x = 0$ for the ring R is prime. It is also aRa faithful, for let $Ra \cdot aya = 0$ then $aRa \cdot aya = 0$, but aRa is a ring without zero divisors, hence $aya = 0$. Condition $(V - W)$ holds, for if $(xa, ay) = xa^2y = 0$ then $(aRxa)(ayRa) = 0$ and as aRa is without zero divisors it follows that $aRxa = 0$ or $ayRa = 0$. Since R is prime it follows that either $xa = 0$ or $ay = 0$. Q.E.D. Condition $(V = W)$ also holds, since $(V, W)v = Ra^2Rxa = 0$, implies for prime rings that either $a^2 = 0$ or $xa = 0$, but $a^2 \neq 0$ since then $(aRa)^2 = 0$ but aRa has no zero divisors, thus $xa = 0$.

A similar result clearly holds for aR.

Applying Theorem 4, we obtain:

Theorem 15. *If R is prime and there exist $a \in R$ such that $d({}_R Ra) < \infty$ and aRa is a ring without zero divisors, then Ra is uniform, $Z({}_R R) = 0$ and R*

is a dense ring of linear transformations of a uniform module over an Öre domain. And if $d(_R R) < \infty$ (which implies $d(_R Ra) < \infty$), then R has a left ring of quotient isomorphic with a total matrix ring over a division ring.

This includes Goldie theorem [2] as we prove:

LEMMA 16. *If a semi prime ring R satisfies the ascending chain condition on left annihilators of the form $(0 : a_1) \subseteq (0 : a_2) \subseteq \cdots$, where $a_{i+1} \in a_i R a_i$ then R contains an element a such that aRa is a ring without zero divisors.*

Proof. Let $0 \neq a \in R$ be such that $(0 : a)$ is maximal; that is, $(0 : a) = (0 : axa)$ or $axa = 0$. First we can assume that $a^2 \neq 0$, indeed the left ideal aR contains an element a_1, such that $a_1{}^2 \neq 0$. If this is not the case then $(x + y)^2 = x^2 + xy + yx + y^2 = xy + yx = 0$ for every $x, y \in R$, but then $[x(aR)]^2 = 0$ since $xyxz = -x^2yz = 0$. But R is semi-prime hence $x(aR) = 0$ for every $x \in aR$, and this shows that $(aR)^2 = 0$ and hence $aR = 0$ which yields $a = 0$. Contradiction.

This element $a_1 = ax_1$ can replace a, and in fact $(0 : a_1) = (0 : a)$. Indeed, clearly $(0 : a) \subseteq (0 : a_1)$. Choose z such that $a_1 za = axza \neq 0$ and then $(0 : a_1) \subseteq (0 : ax_1 za) \subseteq (0 : a)$ which yields $(0 : a_1) = (0 : a)$. In fact, a similar proof shows that $(0 : a_1) = (0 : a_1 y a_1)$ unless $a_1 y a_1 = 0$. So replace a_1 by a and then aRa is a ring without zero divisors. Indeed, if $axa \cdot aya = 0$ and $aya \neq 0$ then $axa \in (0 : aya) = (0 : a)$ and thus $axa^2 = 0$ which means that $ax \in (0 : a^2) \subseteq (0 : a^2ua) = (0 : a)$ if we can find $u \in R$ such that $a^2ua \neq 0$; this is possible since $(a^2R)^2 \neq 0$ and thus $axa = 0$. Q.E.D.

Theorem 15 and Lemma 16, yield Goldie's theorem [2]. Namely,

THEOREM 17A. *If R is prime and satisfies the ascending chain condition on left annihilators, and $d(_R R) < \infty$ then R has an Öre left ring of quotient which is isomorphic with a total matrix ring over a division ring.*

Furthermore, note that the converse is also true, we have by Theorem 15:

THEOREM 17B. *If R is prime, $d(_R R) < \infty$ and R contains an element a such that aRa has no zero divisors then R satisfies the ascending chain condition on left annihilators. Also $d(R_R) = \infty$ or $d(R_R) = d(_R R)$.*

Another result of Theorem 10B is the well known Utumi-Faith theorem (e.g. [5, p. 72]).

V. SEMI-PRIME RINGS

The previous technique of the Morita-context can also be applied to obtain both the result on the structure of semi-simple Artinian rings and the structure of the ring of quotients of semi-prime Goldie's rings [3].

THEOREM 18 (Goldie). *Let R be a semi-prime ring satisfying the ascending chain condition on left annihilator of the form*:

$$(0 : a_1) \subseteq (0 : a_2) \subseteq \cdots, \; a_{i+1} \in a_i R a_i \,, \qquad \textit{then if} \qquad d({}_R R) < \infty$$

— *R has a left ring of quotients which is a finite direct sum of finite dimensional vector spaces over division rings* (= *semi simple artinian*).

In the proof we shall get the classical result:

THEOREM 18A. *If R is semi-prime with the minimum condition on left ideals then the preceding theorem holds and R is equal to its left ring of quotient.*

Proof. We say that a set $a_1, \ldots, a_n$ are orthogonal if $a_i R a_k = 0$ for $i \neq k$. By Lemma 16 it follows also that in the above rings every non zero ideal contains an element a such that $(0 : a)$ is maximal and $a^2 \neq 0$. This enables us to show that the ring R has a maximal set $a_1, \ldots, a_n$ of orthogonal elements for each of which $(0 : a_i)$ is maximal and $a_i^2 \neq 0$. Indeed, choose a_1 any such element which exists by the proof of Lemma 16. Suppose, $a_1, \ldots, a_n$ has been chosen with the above properties and if this is not a maximal set of orthogonal elements then $\{x \mid xRa_i = 0, i = 1, \ldots, n\} = A \neq 0$, and by the preceding result we can find $a_{n+1} \in A$ of the same type and $a_{n+1} R a_i = 0$, $i \neq n + 1$. We also have $a_i R a_{n+1} = 0$ since R is semi-prime and $(a_i R a_{n+1} R)^2 = 0$. This way we can increase the number of the orthogonal elements $a_1, a_2, \ldots$ but this must end since the orthogonality implies that the sum $Ra_1 R \oplus Ra_2 R \oplus \cdots$ is direct and hence must be finite since $d({}_R R) < \infty$. By the preceding it follows also that if $\{a_i\}$ is a maximal orthogonal set then $xRa_i = 0$ for every i implies that $x = 0$, and similarly if $a_i R x = 0$ then $x = 0$.

Let $\{a_1, \ldots, a_n\}$ be a maximal set of the preceding type. Consider the m. c. $M_i = (Ra_i R, Ra_i, a_i R, a_i R a_i)$ which will satisfy Theorem 4. The proof is the same as the proof of Theorem 15, and it remains to show that Ra_i, $a_i R$ are $Ra_i R$-faithful, and $d(Ra_i) < \infty$ as $Ra_i R$ module. Indeed if $x \in Ra_i R$ then $xRa_k = 0$ for $k \neq i$ and also if $xRa_i = 0$ then $x = 0$ by the choice of maximality of the set $\{a_i\}$, which proves that Ra_i is $Ra_i R$ faithful. To prove that $d({}_{Ra_i R} Ra_i) < \infty$ note that if $I_1 \oplus \cdots \oplus I_t$ is a direct sum of $Ra_i R$-left ideals then $Ra_i RI_1 \oplus \cdots \oplus Ra_i RI_t$ is a direct sum of non-zero left R ideal and, therefore, $t \leqslant d({}_R R) < \infty$; a similar argument implies $d(Ra_i R) < \infty$ as a left $Ra_i R$-module. It follows now by Theorem 10B that $T_i = Ra_i R$ has a left ring of quotient $\bar{T}_i$ which is simple Artinian.

The ring of quotients of R is now the same ring of quotients as that of $\sum Ra_i R = T$ which is the direct sum $\sum \bar{T}_i$. To this end, note first that if $t \in T$ is regular in T then it is also regular in R; indeed, if $xt = 0$ $x \in R$ but $Tx \neq 0$ since T has no zero divisor in R, hence $(Tx)t = 0$. As $Tx \subseteq T$ we obtain a contradiction to the regularity of t in T. Furthermore, choose a regular q_i in Ra_iR then $q = \sum q_i$ is regular in $\sum Ra_iR = T$ and, therefore, also in R, since $\sum Ra_iR$ has no zero divisors in R. Now $q \in T$ so also $qR \subseteq T$ and this yields that the ring of quotient $\bar{T}$ of T contains the ring $\{q^{-1}(qr); r \in R\}$ which is isomorphic with R. Clearly, if r is regular in R then qr is in T and regular and, therefore, $(qr)^{-1} q \in \bar{T}$ and from the fact that every element of $\bar{T}$ is of the form $t_1^{-1}t_2$, $t_i \in T \subseteq R$—it follows that $\bar{T}$ is also the ring of left quotient of R.

To prove Theorem 18A, observe that the minimum chain condition on left (or right) ideals implies the maximum condition of left annihilator since if $(0 : a_1) \subseteq (0 : a_2) \subseteq \cdots$ and $a_{i+1} \in a_iRa_i$ then $a_1R^* \supseteq a_2R^* \supseteq \cdots$ and by the minimum condition $a_jR^* = a_kR^*$ for some j and every $k \geqslant j$ but then $(0 : a_j) = (0 : a_jR^*) = (0 : a_kR^*) = (0 : a_k)$ for semi-prime rings. From the previous proof we know that in this case a_iRa_i is a division ring and Ra_iR is a complete ring of linear transformation and so it is its own left ring of quotient. Hence $T = \sum Ra_iR \subseteq R \subseteq \bar{T} = T$. Thus $R = \bar{T} = \bar{R}$ is its own left ring of quotient which proves Theorem 18A. Note also that the minimum condition implies readily that $d({}_RR) < \infty$.

VI. Rings of Endomorphisms

The preceding methods can be extended and applied to the study of the ring of endomorphisms.

Theorem 19. *Let R be a semi-prime ring, with a left Öre ring of quotient R which is semi-simple Artinian. ${}_RV$ be a left R-module with the left module of quotients $\bar{V}$, and $M = (R, V, W, S)$ be a m. c. satisfying*:

(1) $(v, W) = 0$ *implies* $v = 0$;

(2) V_S *is S-faithful.*

Then $\mathscr{E} =$ $\mathrm{Hom}_{\bar{R}}(\bar{V}, \bar{V})$ contains S, and S is dense in $\mathscr{E}$. If $d({}_RV) < \infty$ then S has $\mathscr{E}$ as the Öre left ring of quotient (*compare with* [14]).

Proof. Let V_0 be any finitely generated submodule R-submodule of V, and $\bar{V}_0$ be its module of quotients. Note that by assumption $\bar{V}_0$ is completely reducible. Let $\alpha \in \mathscr{E}$, we want to show that there exists $s \in S$, *regular* on V_0 and $s\alpha \in S$:

Since V_0 is finitely generated, $d({}_{\bar{R}}\bar{V}_0) < \infty$. Furthermore, if $\bar{V}_1 \subseteq \bar{V}_0$ is an irreducible submodule of $\bar{V}_0$ we may write $\bar{V}_1 = \bar{R}v$, where both $v \in V_0$, $v\alpha \in V$. Indeed, let $\bar{V}_1 = \bar{R}\bar{v}$, then $\bar{v} = q_1^{-1}v_1$ $v_1 \in V_0$, $v_1\alpha = q_2^{-1}v_2$, $v_2 \in V$, then $v = q_2v_1 \in V_0$, $v\alpha = q_2v_1\alpha = v_2 \in V$ and $\bar{R}v = \bar{R}\bar{v}$ as required.

We construct the element $s \in S$ as follows:

Choose $0 \neq v_1 \in V_0$, $v_1\alpha \in V$ and such that $\bar{R}v_1$ is irreducible, then $(v_1, W)v_1 \neq 0$, otherwise $(v, W)^2 = 0$ and as R is semi-prime we would have $(v_1, W) = 0$ and, therefore, $v_1 = 0$. Choose $w_1 \in W$, so that $(v_1, w_1)v_1 \neq 0$. Let $L_1 = \{\bar{v} \in \bar{V}_0, \bar{v}s_1 = 0\}$ where $s_1 = [w_1, v_1]$. The sequence $0 \to L_1 \to \bar{V}_0 \xrightarrow{\lambda} \bar{R}v_1 \to 0$ given by $\bar{v}\lambda = \bar{v}[w_1, v_1] = (\bar{v}, w_1)v_1$ is exact and so $\dim({}_{\bar{R}}L_1) = \dim({}_{\bar{R}}\bar{V}_0) - \dim_{\bar{R}}(\bar{R}v_1) = \dim({}_{\bar{R}}\bar{V}_0) - 1$. $\bar{R}v_1 \cap L_1 = 0$ since $v_1 \notin L_1$ and $\bar{R}v_1$ is irreducible by assumption. Hence $\bar{V}_0 = \bar{R}v_1 \oplus L_1$.

Suppose we have chosen $v_1, \dots, v_t$ in V_0, $v_i\alpha \in V$ and $w_1, \dots, w_t \in W$ such that: if we denote $s_j = [w_j, v_j] \neq 0$ $j = 1, \dots, t$ and $L_j = \{\bar{v} \in \bar{V}_0, \bar{v}s_i = 0$ for $i = 1, 2, \dots, j\}$, then $\bar{V}_0 = \bar{R}v_1 \oplus \cdots \oplus \bar{R}v_j \oplus L_j$ $v_js_j \neq 0$ $v_is_j = 0$ for $i < j$. If $L_t \neq 0$, we can continue and choose v_{t+1}, w_{t+1} so that the preceding holds also for $j = t + 1$:

We choose first $v_{t+1} \in L_t$ such that $\bar{R}v_{t+1}$ is irreducible, and $(v_{t+1}, W)v_{t+1} \neq 0$. This can be done since ${}_{\bar{R}}L_t$ is completely reducible and $(v, W)^2 \neq 0$ as before. Note also that we can choose v_{t+1} so that $v_{t+1}\alpha \in V$. Now choose w_{t+1} so that $(v_{t+1}, w_{t+1})v_{t+1} \neq 0$ and then if $s_{t+1} = [w_{t+1}, v_{t+1}]$ we have $v_{t+1}s_{t+1} \neq 0$ and $v_{t+1}s_j = 0$, $j \leqslant t$, since $v_{t+1} \in L_t$. To prove that $\bar{V}_0 = \bar{R}v_1 \oplus \cdots \oplus \bar{R}v_{t+1} \oplus L_{t+1}$, consider the exact sequence

$$0 \to L_{t+1} \to \bar{V}_0 \xrightarrow{\lambda} \bar{R}v_1 \oplus \cdots \oplus \bar{R}v_{t+1} \to 0,$$

where $\lambda : \bar{v} \to \bar{v}s_1 + \cdots + \bar{v}s_{t+1}$ and $L_{t+1} = \mathrm{Ker}(\lambda) = \{\bar{v} \mid \bar{v}s_i = 0, i \leqslant t + 1\}$. It remains to show that λ is onto and that $vs_i \subseteq \bar{R}v_i$. Indeed, if $\bar{v} = q^{-1}v$, $v \in V$ then $vs_i = q^{-1}(v[w_i, v_i]) = q^{-1}(v, w_i)v_i \in \bar{R}v_i$. Each of $\bar{R}v_i$ appears in $\bar{V}_0\lambda$; indeed, $v_{t+1}\lambda = \sum v_{t+1}s_i = v_{t+1}s_{t+1} \neq 0$ and it belongs to $\bar{R}v_{t+1}$ since the latter is irreducible it follows that $\bar{V}_0\lambda$ contains all $\bar{R}v_{t+1}$. Going backward, assume that

$$\bar{V}_0\lambda \supseteq \bar{R}v_{j+1} + \cdots + \bar{R}v_{t+1} \quad \text{then} \quad v_j\lambda = v_js_j + v_{j+1}s_{j+1} + \cdots + v_{t+1}s_{t+1},$$

hence $0 \neq v_js_j \in V_0\lambda$ and so $V_0\lambda \supseteq \bar{R}v_j$. This proves that λ is an epimorphism. We also have $L_{t+1} \cap \sum_{i=1}^{t+1} \bar{R}v_j = 0$. If $\bar{v} = q^{-1} \sum x_iv_i \in L_{t+1}$ then $q\bar{v}s_{t+1} = x_{t+1}v_{t+1}s_{t+1} = 0$ i.e. $x_{t+1}v_{t+1} \in L_{t+1}$. If $x_{t+1}v_{t+1} \neq 0$ then $\bar{R}(x_{t+1}v_{t+1}) = \bar{R}v_{t+1} \subseteq L_{t+1}$ but then $v_{t+1} \in L_{t+1}$ and hence $v_{t+1}s_{t+1} = 0$ which is a contradiction. Similarly, it follows that $\bar{v} = 0$. Thus our induction is completed and we can continue only a finite number of steps, that is as long as $t < \dim(\bar{V}_0)$. So finally we get $\bar{V}_0 = \sum_{i=1}^n \bar{R}v_i, L_n = 0$ and set $s = \sum s_i = \sum [w_i, v_i]$ and $s' = \sum [w_i, v_i\alpha]$. Then $vs\alpha = v \sum [w_i, v_i]\alpha = \sum (v, w_i)v_i\alpha = \sum v[w_i, v_i\alpha] = vs'$, which implies that $s\alpha = s'$. The element s is regular on $\bar{V}_0$, since $\mathrm{Ker}(s) = L_n = \{\bar{v} \mid vs_i = 0, i \leqslant n\} = 0$.

This completes the proof of the density theorem and also the second part if $\dim({}_R\bar{V}) < \infty$.

This result includes Zelmanowitz result [14], since every torsionless module ${}_RV$ is part of a m. c. (R, V, W, S) of Theorem 1, and with $S = \mathrm{Hom}_R(V, V)$.

VII. Morita Context and Radicals

The technique of the m. c. is useful also in studying structure of rings of endomorphisms of torsionless modules. These modules have been shown, in Section I, that they are parts of a m. c. $M = (R, V, W, S)$ where the product $(\ ,\ )$ satisfies: "$(v, W) = 0 \Rightarrow v = 0$". We shall consider a slightly weaker condition.

Theorem 20. *Let $M = (R, V, W, S)$ be a m. c., then $(V, \mathcal{N}(S)W) \subseteq \mathcal{N}(R)$ where $\mathcal{N}(*)$ is one of the following radicals: the lower radical; the locally nilpotent; the Jacobson radical, or the nil radical if the nil radical of R contains all left (or right) nil ideal.*

Proof. Let $\mathcal{N}(*)$ denote the lower radical, and put $\mathcal{N} = \mathcal{N}(S)$. One of the characterizations of the lower radical is as the intersection of all prime ideals, and we use this characterization in our proof. Let P be a prime ideal in R, then $\{s \in S;\ [W, V]s\,[W, V] \subseteq [W, PV]\} = P_S$ is a prime ideal in S; indeed if T_1, T_2 are two ideals in S and $T_1T_2 \subseteq P_S$ then also

$$[W, V]\,T_1[W, V]\,T_2[W, V] \subseteq [W, PV]$$

and hence

$$\begin{aligned}(V, W)(V, T_1W)(VT_2, W)(V, W) &= (V, [W, V]\,T_1[W, V]\,T_2[W, V]W)\\ &\subseteq (V, [W, PV]W)\\ &\subseteq (V, W)\,P(V, W) \subseteq P.\end{aligned}$$

P is prime, hence either $(V, W) \subseteq P$ or (V, T_1W), or (V, T_2W) are contained in P. In the first case $P_S = S$ since $[W, (VW)V] = [W, V]^2$ and so for every $s \in S$ $[WV]\,s[WV] \subseteq P_S$. In each of the other cases $(V, T_iW) \subseteq P$ implies $[W, PV] \supseteq [W, (VT_i, W)V] = [W, V]\,T_i[W, V]$ and hence $T_i \subseteq P_S$. Q.E.D. It follows now that $\mathcal{N}(S) \subseteq P_S$ for every P, that is,

$$[W, V]\,\mathcal{N}(S)[W, V] \subseteq [W, PV]$$

for every prime ideal in P. Thus, as before we get

$$(V, [W, V]\mathcal{N}(S)[W, V]W) = (V, W)(V, \mathcal{N}(S)W)(VW) \subseteq (V, [W, PV]W)$$
$$\subseteq (V, W)P(V, W) \subseteq P,$$

and since P is prime, either $(V, \mathcal{N}(S)W) \subseteq P$ or $(V, W) \subseteq P$ which of course implies that also $(V, \mathcal{N}(S)W) \subseteq P$. This is being true for every prime P, implies $(V, \mathcal{N}(S)W) \subseteq \mathcal{N}(R)$.

Let $\mathcal{N}(*)$ be the locally nilpotent radical. In order to prove that $(V, \mathcal{N}(S)W)$ is locally nilpotent in R it suffices to show that for any finite set of elements $r_i = (v_i, s_i w_i)$, $v_i \in V$, $w_i \in W$ and $s_i \in \mathcal{N}(S)$ there exists an integer m such that every product $r_{i_1} r_{i_2} \cdots r_{i_m} = 0$. Indeed, the set $\{s_i[w_j, v_k]\}$ is a finite set of elements in $\mathcal{N}(S)$ and, therefore, generates a nilpotent ring i.e. any product of m of these elements is zero. Hence,

$$\begin{aligned} r_{i_1} r_{i_2} \cdots r_{i_{m+1}} &= (v_{i_1}, s_{i_1} w_{i_1})(v_{i_2}, s_{i_2} w_{i_2}) \cdots (v_{i_{m+1}}, s_{i_{m+1}} w_{i_{m+1}}) \\ &= (v_{i_1}, s_{i_1}[w_{i_1}, v_{i_2}]\, s_{i_2}[w_{i_2}, v_{i_3}] \cdots s_{i_m}[w_{i_m}, v_{i_{m+1}}]\, s_{i_{m+1}} v_{i_{m+1}}) \\ &= 0. \end{aligned}$$

Here we used repeatedly the relation:

$$(v, sw)(v', s'w') = (v, sw(v', s'w')) = (v, s[w, v']\, s'w').$$

This concludes the proof for the locally nilpotent radical.

A similar application of the last relation yield the nil case: For let $\mathcal{N}(*)$ be the nil radical and $r = (v, sw) \in (V, \mathcal{N}(S)W)$, $s \in \mathcal{N}(S)$ then $[w, v]s$ is nil for every fixed $[w, v]$. Hence:

$$\begin{aligned} (v, sw)^{k+1} &= (v, sw)(v, sw)(v, sw)^{k-2} = (v, s[w, v]\, sw)(v, sw)^{k-2} \\ &= \cdots = (v, s([w, v]s)^k w) = 0 \end{aligned}$$

and so (V, sw) is a nil left ideal in R and by our assumption $(V, sw) \subseteq \mathcal{N}(R)$ and so $(V, \mathcal{N}(S)W) \subseteq \mathcal{N}(R)$.

Let $\mathcal{N}(*)$ be the Jacobson radical: Let $(v, sw) \in (V, \mathcal{N}(S)W)$ with $s \in \mathcal{N}(S)$. Again for fixed v, w, $s[w, v]$ has a left quasi inverse s', that is $s[w, v] + s' - s's[w, v] = 0$ and so $s' = t[w, v]$. This yields the relation: $(s + t - t[w, v]s)[w, v] = 0$. Consider the element,

$$r = (v, sw) + (v, tw) - (v, tw)(v, sw) = (v, sw + tw - t[w, v]\, sw),$$

which will satisfy

$$rv = (v, sw + tw - t[w, v]\, sw)v = v(s + t - t[w, v]s)[w, v] = 0,$$

and therefore $r^2 = r(v, *) = (rv, *) = 0$, which proves that (v, sw) has a left quasi inverse, namely $(-r) \circ (v, tw) = -r + (v, tw) + r(v, tw)$. Since $0 = (-r) \circ r = (-r) \circ (v, tw) \circ (v, sw)$.

We can now conclude that, (V, sw) for fixed w and $s \in \mathscr{N}(S)$ is a left quasi regular ideal hence $(V, sw) \subseteq \mathscr{N}(R)$, and being true for all $w \in W$ we get $(V, \mathscr{N}(S)W) \subseteq \mathscr{N}(R)$. Q.E.D.

The preceding theorem yields immediately:

COROLLARY 21. *If $M = (R, V, W, S)$ satisfies the condition*:

(A) $(Vs, W) = (V, sW) = 0$ *implies* $s = 0$,

then if R is either semi prime; *without locally nilpotent ideals*; *semi-simple in the sense of Jacobson—then so is the ring S. If R has no nil lift ideals then also S has this property.*

This is a simple consequence of the fact that we have $(V, \mathscr{N}(S)W) \subseteq \mathscr{N}(R) = 0$. The last assertion follows from a more detailed study of our proof of the nil case which actually shows that $\mathscr{N}$ is a left nil ideal then $(V, sw) \subseteq \mathscr{N}(R)$ for every $s \in \mathscr{N}$, $w \in W$ which in the present case yields the same relation $(V, \mathscr{N}(S)W) = 0$ and so $\mathscr{N}(S) = 0$.

Condition (A) of our last corollary is not recognized easily: We encounter more often (e.g. the case of torsionless modules) the following which implies (A).

LEMMA 22. *The m. c. $M = (R, V, W, S)$ satisfies* (A)—*if one of the following holds in M*:

(A_1) $(V, w) = 0$ *implies* $w = 0$ *and* ${}_SW$ *is faithful*;

(A_2) $(v, W) = 0$ *implies* $v = 0$ *and* V_S *is faithful.*

Note that (A_1) holds by definition for all torsionless modules.

Proof. If $(V, sW) = (Vs, W) = 0$ then by (A_1) $sW = 0$ and so $s = 0$ and if (A_2) holds $Vs = 0$ and so $s = 0$. Q.E.D.

By symmetry, it follows that $M^* = (S, W, V, R)$ is also a m. c. with [ ], ( ) interchangable, hence Theorem 19 yields:

COROLLARY 23. *In each of the cases of Theorem* 19 (*with the respective condition on nil radicals*) *we have* $[W, \mathscr{N}(R)V] \subseteq \mathscr{N}(S)$.

Can we characterize the radical of S by V, W and $\mathscr{N}(R)$. We can only show the following

LEMMA 24. *In each of the cases of Theorem* 20, $(V, sW) \subseteq \mathcal{N}(R)$ *if and only if* $[W, V]\, s[W, V] \subseteq \mathcal{N}(S)$.

Proof. If $(V, sW) \subseteq \mathcal{N}(R)$ then

$$\mathcal{N}(S) \supseteq [W, \mathcal{N}(R)V] \supseteq [W, (V, sW)V] = [W, Vs[W, V]] = [W, V]\, s[W, V].$$

Conversely, if $\mathcal{N}(S) \supseteq [W, V]\, s[W, V]$ then

$$\begin{aligned}\mathcal{N}(R) &\supseteq (V, \mathcal{N}(S)W) \supseteq (V, [W, V]s\, [WV]W) \supseteq (V, W)(V, sW)(V, W)\\ &\supseteq (V, sW)^3.\end{aligned}$$

In each of the cases $R/\mathcal{N}(R)$ has no nilpotent ideals, hence $(V, sW) \subseteq \mathcal{N}(R)$.

In particular, it follows immediately

COROLLARY 25. *If* $[W, V] = S$ *then in each of the cases of Theorem* 20 $\mathcal{N}(S) = \{s \mid (v_i, sw_k) \in \mathcal{N}(R)\}$ *for a set of R-generators* $\{v_i\}$ *of V and* $\{w_k\}$ *of W.*

Under the conditions of the theorem, it follows that $(V, sW) \subseteq \mathcal{N}(R)$ is equivalent, in view of Lemma 24, to $SsS \subseteq \mathcal{N}(S)$ and hence $s \in \mathcal{N}(S)$.

A classical case where the conditions of the last corollary holds is for ${}_RV$ finitely generated projective and $M = (R, V, \mathrm{Hom}_R(V, R), \mathrm{Hom}_R(V, V))$ is the standard m. c. ([13]). Then if $1 = \sum_{i=1}^{n} [w_i, v_i]$, v_i are a set of generators of V and w_i a set of generators of W, since $1W = \sum [w_i, v_i]W \subseteq \sum w_i R = W$ and $V1 = \sum V[w_i, v_i] \subseteq \sum Rv_i = V$. Hence:

COROLLARY 26. *If* ${}_RV$ *is finitely generated projective and* $1 = \sum [\varphi_i, v_i]$ *in* $\mathcal{E} = \mathrm{Hom}_R(V, V)$, $\varphi_i \in \mathrm{Hom}_R(V, R)$. *Then* $\alpha \in \mathcal{E}$ *belongs to* $\mathcal{N}(\mathcal{E})$, *in one of the radicals of Theorem* 19, *if and only if* $(v_i, \alpha\varphi_i) \in \mathcal{N}(R)$.

This includes the result that $\mathcal{N}(R_n) = \mathcal{N}(R)_n$ by taking V to be a free R-module on n-generators, and then $\mathcal{E} = R_n$, and if φ_i are the dual base then $(v_i, \alpha\varphi_k)$ are the (i, k) entries of R_n.

VIII. PRIME AND PRIMITIVE RINGS

THEOREM 27. *Let* $M = (R, V, W, S)$ *be a m. c. satisfying the preceding condition* (c): "$(V, sW) = (Vs, W) = 0 \Rightarrow s = 0$"—*then if R is prime or left primitive then so is S.*

Proof. Let R be prime, and let $sSt = 0$ then $s[W, V]t = 0$ and therefore $(V, sW)(Vt, W) = (V, sW(Vt, W)) = (V, s[W, V]\, tW) = 0$. R is prime and both (V, sW) and (Vt, W) are ideals in R, hence either $(V, sW) = 0$ or (Vt, W) which yields either $s = 0$ or $t = 0$. Q.E.D.

Let R be left primitive, ${}_RL$ be a maximal modular left ideal in R, then R/L is an irreducible faithful module and so $rR \subseteq L$ implies $r = 0$. Consider now the module $W_0 = \{w \in W, (V, w) \subseteq L\}$. The module W_0 is a left S-module since $(V, sw) = (Vs, w) \subseteq (V, w) \subseteq L$. Next the module $\overline{W} = (WR + W_0)/W_0$ is a left irreducible and faithful S-module: Indeed, ${}_S\overline{W}$ is faithful for if $s(WR) \subseteq W_0$ then $(V, sWR) \subseteq L$ and so $(V, sW)R \subseteq L$ and consequently $(V, sW) = 0$ and hence $s = 0$ by (c). Next ${}_S\overline{W}$ is irreducible for let $0 \neq \overline{w} \in \overline{W}$ be represented by w, then $w \notin W_0$ and hence $(V, w) \not\subseteq L$. Consequently, by the maximality of L it follows that $(V, w) + L = R$, and hence $WR = W(V, w) + WL = [W, V]w + WL \subseteq Sw + WL$. Now $WL \subseteq W_0$ as $(V, WL) \subseteq (V, W)L \subseteq L$ and thus $S\overline{w} = \overline{WR} = \overline{W}$. i.e. ${}_S\overline{W}$ is irreducible.

Q.E.D.

Finally, $\overline{W} \neq 0$, since $\overline{W} = 0$ will mean $WR \subseteq W_0$ and so $(V, W)R = (V, WR) \subseteq (V, W_0) \subseteq L$ and thus $(V, W) = 0$ which is impossible.

In particular, the last theorem yields, in view of the fact that torsionless modules satisfy Lemma 21 and therefore also condition (c):

COROLLARY 28. *Let ${}_RV$ be faithful and torsionless (e.g. ${}_RV$ be free) then if R is either prime or left primitive then so is* $\mathrm{Hom}_R(V, V)$.

The prime case was proved in [14], and the primitive case for ${}_RV$ finitely generated and free was proved in [4] and [10].

REFERENCES

1. H. BASS, The Morita-theorems, Oregon Lectures, 1962.
2. A. W. GOLDIE, The structure of prime rings under ascending chain conditions, *Proc. London Math. Soc.* **8** (1958), 589–608.
3. A. W. GOLDIE, Semi-prime rings with maximum condition, *Proc. London Math. Soc.* **10** (1960), 201–220.
4. R. N. GUPTA, On primitivity of matrix rings, *Amer. Math. Monthly* **75** (1968), 636.
5. I. N. HERSTEIN, Topics in Ring Theory, *in* "Chicago Lectures in Math.", The University of Chicago Press, 1969.
6. N. JACOBSON, Structure of rings, *Amer. Math. Soc. Colloq. Publ.* **37** (1964).
7. R. E. JOHNSON, The extended centralizer of a ring over a module, *Proc. Amer. Math. Soc.* **2** (1951), 891–895.
8. R. E. JOHNSON, Quotient rings of rings with zero singular ideal. *Pacific J. Math.* **11** (1961), 1385–1389.
9. K. MORITA, Duality for modules and its application to the theory of rings with minimum conditions, *Science Reports of the Tokyo Kyoiku Daigoku Sect. A.* **6** (1958), 83–142.
10. E. C. POSNER, Primitive matrix rings, *Arch. Math.* **12** (1961), 97–107.
11. Y. UTUMI, On quotient rings, *Osaka Math. J.* **8** (1956), 1–18.

12. Y. Utumi, On rings of which any one-sided quotient rings are two-sided, *Proc. Amer. Math. Soc.* **14** (1963), 141–147.
13. E. R. Willard, Properties of projective generators, *Math. Ann.* **158** (1965), 352–364.
14. J. M. Zelmanowitz, Endomorphism rings of torsionless modules, *J. Algebra* **5** (1967), 325–341.